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RIGIDITY OF p-COMPLETED CLASSIFYING SPACES OF ALTERNATING GROUPS AND CLASSICAL GROUPS OVER A FINITE FIELD

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ABSTRACT. A p-adic rigid structure of the classifying spaces of certain finite groups π, including alternating groups A_n and finite classical groups, is shown in terms of the maps into the p-completed classifying spaces of compact Lie groups. The spaces $(B\pi)^{\wedge}_p$ have no nontrivial retracts. As an application, it is shown that $(BA_n)^{\wedge}_p \simeq (B\Sigma_n)^{\wedge}_p$ if and only if $n \not\equiv 0, 1, \bmod p$. It is also shown that $(BSL(n, \mathbb{F}_q))^{\wedge}_p \simeq (BGL(n, \mathbb{F}_q))^{\wedge}_p$ where q is a power of p if and only if $(n, q-1) = 1$.

If K and G are compact Lie groups, there are usually relatively few homotopy classes of maps $BK \to BG$ or $(BK)^{\wedge}_p \to (BG)^{\wedge}_p$. For instance, if K is connected and simple and G is connected with $\operatorname{rank}(K) > \operatorname{rank}(G)$, the homotopy sets $[BK, BG]$ and $[(BK)^{\wedge}_p, (BG)^{\wedge}_p]$ are trivial [1, 20] and the p-completion $(BK)^{\wedge}_p$ has no nontrivial retracts at any prime p [11]. We will prove similar results with $(BK)^{\wedge}_p$ replaced by $(B\pi)^{\wedge}_p$, where π is an alternating group or a classical group over a finite field, and the notion of rank replaced by the notion of p-rank. (The p-rank of a group π is the maximal rank of an elementary abelian p-subgroup of π.)

Let G be a compact Lie group. Recall that if π is a finite group with p-Sylow subgroup π_p and f is a map $(B\pi)^{\wedge}_p \to (BG)^{\wedge}_p$, then the restriction $f|B\pi_p$ must be of the form $B\rho$ for some homomorphism $\rho: \pi_p \to G$, [7, 2, 15]. The following theorem gives a sufficient condition that the homomorphism ρ be one-to-one in terms of weak closures of elements of the center of the p-Sylow subgroup π_p. The weak closure of the one-element set $\{z\}$ in π_p with respect to π is the subgroup of the p-Sylow subgroup generated by the set $\{xzx^{-1} | x \in \pi\} \cap \pi_p$. We prove

Theorem 1. *Let π be a finite group with p-Sylow subgroup π_p. Suppose that, for any nonidentity element z of the center of π_p, the weak closure of $\{z\}$ in π_p with respect to π is equal to the p-Sylow subgroup π_p. If G is a compact Lie group and $f: (B\pi)^{\wedge}_p \to (BG)^{\wedge}_p$ is a nonzero map with $f|B\pi_p \simeq B\rho$ for a homomorphism ρ, then $\rho: \pi_p \to G$ is injective.*

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Corollary 2. *Let π be a finite group with p-Sylow subgroup π_p and let G be a compact Lie group. Assume that if $f\colon (B\pi)^\wedge_p \to (BG)^\wedge_p$ is a nonzero map with $f|B\pi_p \simeq B\rho$, then the homomorphism $\rho : \pi_p \to G$ is injective. Then each of the following holds:*

(a) *If p-$\mathrm{rank}(\pi) > p$-$\mathrm{rank}(G)$, then $[(B\pi)^\wedge_p, (BG)^\wedge_p] = 0$, and the evaluation map $\mathrm{map}((B\pi)^\wedge_p, (BG)^\wedge_p) \to (BG)^\wedge_p$ is a weak equivalence.*

(b) *The p-complete classifying space $(B\pi)^\wedge_p$ has no nontrivial retracts.*

We will show that the hypothesis of π in Theorem 1 is satisfied by many finite (simple) groups at p. The list of such groups contains the alternating groups A_n at any prime p, the finite classical groups $GL(n, \mathbb{F}_q)$, $O(n, \mathbb{F}_q)$ with $n \geq 5$ and q odd, $Sp(2n, \mathbb{F}_q)$ with $(n, q) \neq (2, 2)$ and $U(2n, \mathbb{F}_{q^2})$ at p which is the characteristic of the finite fields. All of the finite simple groups of types A, B, C, and D associated with the above classical groups at the prime p also satisfy the hypothesis in Theorem 1. Consequently Theorem 1 and Corollary 2 hold for these groups.

The proof of Theorem 1 makes use of the property of the images of conjugacy classes in a p-Sylow subgroup under the homomorphism ρ. This property is stated in Lemma 1.1. Since $f|B\pi_p \simeq 0$ implies $f \simeq 0$ [9], the remaining work is to show that if ρ is not injective, the homomorphism is trivial. A sufficient condition is the hypothesis dealing with weak closures.

This hypothesis is related to the fusion problem in group theory, [19]. G. Glauberman points out that there are finite simple groups which do not satisfy the hypothesis. An example is given by the projective group $PSU(3, \mathbb{F}_{p^2})$ of 3×3 special unitary matrices with p odd, since the center of a p-Sylow subgroup is strongly closed. One can show, however, that Corollary 2 holds for this group at the prime p. This suggests that Corollary 2 may be true without assumption of the property of a p-Sylow subgroup if the finite group π is simple.

This paper consists of six sections. In §1, we discuss maps between classifying spaces and prove Theorem 1 and Corollary 2. From §2 to §6, alternating groups, symmetric groups, and finite classical groups are treated. In particular, we show that the hypothesis of π in Theorem 1 is satisfied by these groups at a suitable prime p.

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1. Maps between classifying spaces

Suppose H is a subgroup of G, and $x, y \in H$. We say that x and y are conjugate in G, denoted by $x \underset{G}{\sim} y$, if $y = gxg^{-1}$ for some $g \in G$. For $g \in G$, the conjugation map $C_g : H \to gHg^{-1}$ is defined by $C_g(x) = gxg^{-1}$ for each $x \in H$. If $g \in H$, then the self-map BC_g of BH is homotopic to the identity map, [18].

Lemma 1.1. *Suppose a finite p-group γ is a subgroup of a compact Lie group G'. Let $f : (BG')^\wedge_p \to (BG)^\wedge_p$ with $f|B\gamma \simeq B\rho$ for $\rho \in \mathrm{Hom}(\gamma, G)$. If $x, y \in \gamma$ and $x \underset{G'}{\sim} y$, then $\rho(x) \underset{G}{\sim} \rho(y)$.*

Proof. Suppose $y = uxu^{-1}$ for $u \in G'$. Since the conjugation map BC_u on $(BG')^\wedge_p$ is homotopic to the identity, we have the homotopy commutative diagram:

$$\begin{array}{ccc} B\gamma & \xrightarrow{Bj_1} & (BG')^\wedge_p f \searrow \\ {\scriptstyle BC_u}\downarrow & & {\scriptstyle BC_u}\downarrow \qquad (BG)^\wedge_p\,. \\ B(u\gamma u^{-1}) & \xrightarrow{Bj_2} & (BG')^\wedge_p f \nearrow \end{array}$$

In this diagram j_1 and j_2 are the inclusions. We see $f \circ Bj_2 \simeq B\rho'$ for some $\rho' \in \mathrm{Hom}(u\gamma u^{-1}, G)$. Since $f \circ Bj_1 \simeq f \circ Bj_2 \circ BC_u$, it follows that $\rho = \rho' \circ C_u$ up to G-conjugation. Consequently $\rho(x) \underset{G}{\sim} \rho' \circ C_u(x) = \rho'(y)$. Next suppose $i_1 : \gamma \cap u\gamma u^{-1} \to \gamma$ and $i_2 : \gamma \cap u\gamma u^{-1} \to u\gamma u^{-1}$ are the inclusions. We notice that $y \in \gamma \cap u\gamma u^{-1}$ and $f \circ Bj_1 \circ Bi_1 \simeq f \circ Bj_2 \circ Bi_2$. It follows that $\rho(y) \underset{G}{\sim} \rho'(y)$ and therefore $\rho(x) \underset{G}{\sim} \rho(y)$. □

We remark here that if $x \in \ker \rho$ and $x \underset{G'}{\sim} y$, then $y \in \ker \rho$. Consequently the weak closure of any subset of $\ker \rho$ in γ with respect to G' is included in $\ker \rho$.

Lemma 1.2. *Suppose f is a map from $(B\pi)^\wedge_p$ to $(BG)^\wedge_p$. If $f|B\pi_p \simeq 0$, then $f \simeq 0$.*

Proof. Along the line of the proof of result of Friedlander-Mislin [9, Theorem 3.1] we see that if the component of the mapping space $\mathrm{map}_*(B\gamma, (BG)^\wedge_p)_0$ which contains the constant map is weakly contractible for any finite p-group γ, then the map $f: (B\pi)^\wedge_p \to (BG)^\wedge_p$ factors through a p-cyclic space defined in [13]. From the fibration $\mathrm{map}_*(X, Y)_0 \to \mathrm{map}(X, Y)_0 \to Y$, we see that $\mathrm{map}_*(X, Y)_0$ is weakly contractible if and only if the basepoint evaluation map $\varepsilon : \mathrm{map}(X, Y)_0 \to Y$ is a weak equivalence. Suppose $\lambda : Y \to \mathrm{map}(X, Y)_0$ is the map which sends $y \in Y$ to the constant map $\lambda(y)(x) = y$. Note here that the composite $\varepsilon \circ \lambda$ is the identity map. Consequently if λ is an equivalence, so is ε. To complete the proof, it remains to show that the map $\lambda : (BG)^\wedge_p \to \mathrm{map}(B\gamma, (BG)^\wedge_p)_0$ is weakly equivalent. We use an induction on the order of the finite p-group γ. A result of Lannes [12] implies the case for $\gamma = \mathbb{Z}/p$, since G is the centralizer of the trivial homomorphism $\gamma \to G$. In general, consider a group extension $1 \to N \to \gamma \to \sigma \to 1$ where $\sigma = \mathbb{Z}/p$. Recall that the homotopy fixed point space $\mathrm{map}_\sigma(E\sigma, \mathrm{map}(BN, (BG)^\wedge_p))$ is homotopy equivalent to $\mathrm{map}(B\gamma, (BG)^\wedge_p)$. The σ-action on $\mathrm{map}(BN, (BG)^\wedge_p) = \mathrm{map}_N(E\gamma, (BG)^\wedge_p)$ is given by the rule $(f \cdot s)(u) = f(ur^{-1}) \cdot r$ where $f \in \mathrm{map}_N(E\gamma, (BG)^\wedge_p)$, $s \in \sigma$ and $r \in \gamma$ is a preimage of s under the epimorphism $\gamma \to \sigma$. Consequently, one has the commutative diagram:

$$\begin{array}{ccc} (BG)^\wedge_p & \xrightarrow{\lambda} & \mathrm{map}(B\gamma, (BG)^\wedge_p)_0 \\ \downarrow{\scriptstyle \lambda_\sigma} & & \uparrow \\ \mathrm{map}(B\sigma, (BG)^\wedge_p)_0 & \longrightarrow & \mathrm{map}_\sigma(E\sigma, \mathrm{map}(BN, (BG)^\wedge_p)_0)_0 \end{array}$$

Since the vertical maps are homotopy equivalences, it remains to show the lower horizontal map is an equivalence. This map is induced by the σ-equivalence

$(BG)^\wedge_p \xrightarrow{\lambda_N} \mathrm{map}(BN, X)_0$, where the action of σ on the space $(BG)^\wedge_p$ is trivial. We conclude that λ is a homotopy equivalence. $\square$

The result of Friedlander-Mislin implies that Lemma 1.2 is still true when $(BG)^\wedge_p$ is replaced by the p-completion of a simply connected space whose loop space is homotopy equivalent to a finite dimensional complex. In fact, we have seen that the result holds for a space X if $\mathrm{map}_*(B\gamma, X)_0$ is weakly contractible for any finite p-group γ. This condition is satisfied by a simply connected p-complete space X where $H^*(X; \mathbb{F}_p)$ is finitely generated as an algebra. This is due to Dwyer-Wilkerson [6].

Proof of Theorem 1. From Lemma 1.2, it suffices to show that if $f|B\pi_p \simeq B\rho$ and the homomorphism $\rho : \pi_p \to G$ is not injective, then ρ is trivial. Suppose $\ker\rho \neq 1$. Then $\ker\rho$ is a nontrivial normal subgroup of a finite p-group. Hence $\ker\rho$ must contain a nonidentity element of the center of π_p. Lemma 1.1 together with our assumption shows $\ker\rho = \pi_p$. $\square$

Lemma 1.3. *The evaluation map* $\mathrm{map}((B\pi)^\wedge_p, (BG)^\wedge_p)_0 \to (BG)^\wedge_p$ *is weakly equivalent.*

Proof. It suffices to show the fibre $\mathrm{map}_*((B\pi)^\wedge_p, (BG)^\wedge_p)_0$ is weakly contractible. Recall that the natural map

$$\operatorname*{hocolim}_{\pi/\pi_\alpha \in O_p(\pi)} (E\pi \times_\pi \pi/\pi_\alpha) \to B\pi$$

is a mod p homology isomorphism, [14, Lemma 3.1]. Here π_α is a p-subgroup of π. Since $\pi_1(BG)$ is finite, the space BG is $\mathbb{Z}/p$-good [2, p. 215]. Consequently

$$\pi_i \mathrm{map}_*(X, (BG)^\wedge_p) = \pi_i \mathrm{map}_*(\widehat{X}_p, (BG)^\wedge_p)$$

for any X and any $i \geq 0$. Hence we see the following:

$$\begin{aligned}
&\pi_i \mathrm{map}_*((B\pi)^\wedge_p, (BG)^\wedge_p)_0 \\
&= \pi_i \mathrm{map}_* \left(\left(\operatorname*{hocolim}_{\vec{\alpha}} B\pi_\alpha \right)^\wedge_p, (BG)^\wedge_p \right)_0 \\
&= \pi_i \mathrm{map}_* \left(\operatorname*{hocolim}_{\vec{\alpha}} B\pi_\alpha, (BG)^\wedge_p \right)_0 \\
&= \pi_i \operatorname*{holim}_{\overleftarrow{\alpha}} \mathrm{map}_*(B\pi_\alpha, (BG)^\wedge_p)_0 .
\end{aligned}$$

In the proof of Lemma 1.2, we have seen that $\mathrm{map}_*(B\pi_\alpha, (BG)^\wedge_p)$ is weakly contractible for any α. Consequently, so is the cosimplicial replacement $\prod^* \{\mathrm{map}_*(B\pi_\alpha, (BG)^\wedge_p)\}$, [2, p. 303]. By [2, Mapping Lemma, p. 285] we see that

$$\operatorname*{holim}_{\overleftarrow{\alpha}} \mathrm{map}_*(B\pi_\alpha, (BG)^\wedge_p)_0$$

is weakly contractible and hence so is $\mathrm{map}_*((B\pi)^\wedge_p, (BG)^\wedge_p)_0$. $\square$

Lemma 1.4. *Let f be a self-map of $(B\pi)^\wedge_p$ with $f|B\pi_p \simeq B\rho$. The map f is a homotopy equivalence if and only if the homomorphism ρ is injective.*

Proof. Suppose f is a homotopy equivalence. If ρ is not injective, then we can find a subgroup $\mathbb{Z}/p$ of $\ker\rho$. Let $i\colon \mathbb{Z}/p \to \pi_p$ and $j\colon \pi_p \to \pi$ be the inclusions. We consider the commutative diagram

$$\begin{array}{ccc}
H^*(B\pi;\mathbb{F}_p) & \xleftarrow[f^*]{} & H^*(B\pi;\mathbb{F}_p) \\
\big\downarrow{\scriptstyle Bj^*} & \swarrow{\scriptstyle B\rho^*} & \\
H^*(B\pi_p;\mathbb{F}_p) & & \\
\big\downarrow{\scriptstyle Bi^*} & & \\
H^*(B\mathbb{Z}/p;\mathbb{F}_p) & &
\end{array}$$

Since $\mathbb{Z}/p \subset \ker\rho$, we see that $Bi^* \circ B\rho^* = B(\rho\cdot i)^* = 0$. On the other hand, a result of Lannes [12] implies that the natural map $[B\mathbb{Z}/p, (B\pi)^\wedge_p] \to \mathrm{Hom}_{\mathscr{A}_p}(H^*B\pi, H^*B\mathbb{Z}/p)$ is bijective. Consequently $Bi^* \circ Bj^* \neq 0$. Since f is a homotopy equivalence, the map f^* must be an isomorphism. Thus the composition $Bi^* \circ Bj^* \circ f^*$ would not be zero. This is a contradiction, since this composition is equal to the zero map $Bi^* \circ B\rho^*$.

Next suppose ρ is injective. Then $B\rho^*$ is injective by transfer argument. Hence the self-map f^* is injective on each finite dimensional vector space $H^n(B\pi;\mathbb{F}_p)$ and hence f^* is bijective for dimensional reasons. Therefore the self-map f of $(B\pi)^\wedge_p$ is a homotopy equivalence. □

Proof of Corollary 2. (a) If p-$\mathrm{rank}(\pi) > p$-$\mathrm{rank}(G)$ and $f\colon (B\pi)^\wedge_p \to (BG)^\wedge_p$ with $f|B\pi_p \simeq B\rho$, then Theorem 1 shows $f \simeq 0$. Thus $\mathrm{map}((B\pi)^\wedge_p, (BG)^\wedge_p) = \mathrm{map}((B\pi)^\wedge_p, (BG)^\wedge_p)_0$. The desired result follows from Lemma 1.3.

(b) Suppose X is a nontrivial retract of $(B\pi)^\wedge_p$ with $X \underset{r}{\overset{i}{\leftrightarrows}} (B\pi)^\wedge_p$ and $r\circ i \simeq 1_X$. If $i\circ r|_{b\pi_p} \simeq B\rho$, then Theorem 1 says that ρ is injective. Lemma 1.4 shows $i\circ r$ is a homotopy equivalence. Consequently, the epimorphism i^* is also a monomorphism. Hence the map i would be a homotopy equivalence. This contradiction completes the proof. □

2. Alternating groups and symmetric groups

We will prove that the alternating group A_n satisfies the hypothesis of Theorem 1. To do so we need to take a close look at the center of a p-Sylow subgroup. The following lemma will be used for A_n and other finite groups.

Lemma 2.1. *Suppose $V \rtimes H$ is a semidirect product where the center of H, denoted by $Z(H)$, acts faithfully on the abelian group V. Then the center of the group $V \rtimes H$ is equal to the set $\{v \in V | hv = vh$ for any $h \in H\}$.*

Proof. It is clear that $Z(V\rtimes H)$ includes this set since V is abelian Conversely, if $v_0h_0 \in Z(V \rtimes H)$ where $v_0 \in V$ and $h_0 \in H$, then we have

$$(v_0h_0)h = v_0 \cdot h_0h, \quad h(v_0h_0) = hv_0h^{-1}\cdot hh_0.$$

Hence $hv_0h^{-1} = v_0$ and $h_0h = hh_0$ for any $h \in H$. We note that $h_0 \in Z(H)$. It remains to show $h_0 = 1$. Consider the following

$$(v_0h_0)\cdot v = v_0h_0vh_0^{-1}\cdot h_0, \quad v(v_0h_0) = vv_0\cdot h_0.$$

Since V is abelian, we see $h_0 v h_0^{-1} = v$ for any $v \in V$. According to our assumption, $Z(H)$ acts faithfully on V. Consequently $h_0 = 1$, since $h_0 \in Z(H)$. □

Proposition 2.2. *The alternating group* A_n *satisfies the hypothesis of Theorem* 1 *at any prime* p.

Proof. Since any two p-Sylow subgroups are conjugate, we may choose one to prove the desired result.

First suppose p is odd. Then a p-Sylow subgroup of A_n is also a p-Sylow subgroup of the symmetric group Σ_n. If n is a power of p, a p-Sylow subgroup of A_{p^i}, say P_i can be expressed as the wreath product $P_{i-1} \wr C_p$ where C_p is a cyclic group of order p and $P_1 = \mathbb{Z}/p\langle(1, 2 \cdots p)\rangle$. For example, if $i = 2$, the cyclic group C_p is generated by $(1\ p+1 \cdots (p-1)p+1) \cdots (p-1)p+(p-1) \cdots (p-1)p+(p-1))$. In general, if $n = a_0 + a_1 p + \cdots + a_k p^k$ with $0 \le a_i < p$ for $i = 0, \ldots, k$, then $\prod_{i=1}^{k}(\prod^{a_i} P_i)$ is a p-Sylow subgroup of A_n.

When $n < 2p$, the p-Sylow subgroup is isomorphic to $\mathbb{Z}/p$ if it is not trivial. Obviously the result holds. We now assume $2p \le n$. Since

$$Z\left(\prod_{i=1}^{k}\prod^{a_i} P_i\right) = \prod_{i=1}^{k}\prod^{a_i} Z(P_i),$$

it suffices to consider the case $n = p^i$ for some $i \ge 2$. Using Lemma 2.1 one can show that (for any p) the center of P_i is isomorphic to $\mathbb{Z}/p$ generated by $z = (1 \cdots p)(p+1 \cdots 2p) \cdots (p^i - p + 1 \cdots p^i)$. Let $z' = (1 \cdots p)(p+1 \cdots 2p)^{-1} \cdots (p^i - p + 1 \cdots p^i)^{-1}$ so that $z \cdot z' = (1 \cdots p)^2$ and $z' \in P_i$. Note here that P_i contains all of the above p-cycles. We claim $z \underset{A_{p^i}}{\sim} z'$. Notice that if σ is a p-cycle, then σ^{-1} is also a p-cycle. Hence there is a $g \in \Sigma_p$ such that $\sigma^{-1} = g\sigma g^{-1}$. Consequently we can find $\hat{g} \in \Sigma_p \times \cdots \times \Sigma_p \subset \Sigma_{p^i}$ such that $z' = \hat{g} z \hat{g}^{-1}$. If $\hat{g} \in A_{pi}$, we are done. If $\hat{g} \notin A_{p^i}$, let

$$h = (1 \quad p+1)(2 \quad p+2) \cdots (p \quad 2p).$$

The conjugation by h switches the first p-cycle and the second one. If $\hat{g} = 1 \times g_2 \times \cdots \times g_k \in \Sigma_p \times \cdots \times \Sigma_p$, let $\overline{g} = g_2 \times 1 \times g_3 \times \cdots \times g_k$. It follows that $\overline{g}h \in A_{p^i}$ for $i \ge 2$ and that $(\overline{g}h)z(\overline{g}h)^{-1} = z'$. We now see that the weak closure of $\{z\}$ contains a p-cycle since p is odd. Consequently we can show that any generator of P_i is conjugate in A_{p_i} to an element of the elementary p-abelian subgroup of P_i generated by the p-cycles $(1 \cdots p), \ldots, (p^i - p + 1 \cdots p^i)$. Thus the weak closure is equal to the p-Sylow subgroup P_i.

Next suppose $p = 2$. The argument is similar to the one we just used. Let $A_n(2)$ and $\Sigma_n(2)$ denote a 2-Sylow subgroup of A_n and that of Σ_n respectively. We first consider when n is a power of 2. For example, one sees that $A_4(2)$ is $\mathbb{Z}/2 \times \mathbb{Z}/2$ generated by $(12)(34)$ and $(13)(24)$. For $n = 8$, if

$$E = \mathbb{Z}/2\langle(12)(34)\rangle \times \mathbb{Z}/2\langle(12)(56)\rangle \times \mathbb{Z}/2\langle(12)(78)\rangle$$

and

$$K = (\mathbb{Z}/2\langle(13)(24)\rangle \times \mathbb{Z}/2\langle(57)(68)\rangle) \rtimes \mathbb{Z}/2\langle\sigma\rangle$$

where $\sigma = (15)(26)(37)(48)$, then K normalizes E and $A_8(2) = E \cdot K$. Inductively $A_{2^i}(2) = E_i \cdot K_i$ where E_i is an elementary abelian 2-group and K_i

is a group generated by $K_{i-1} \times K_{i-1}$ and the element $(1\ 2^{i-1}+1)\cdots(2^{i-1}\ 2^i)$. Again using Lemma 2.1 one can show that the center of $A_{2^i}(2)$ is isomorphic to $\mathbb{Z}/2$ generated by

$$Z = (12)(34)(56)(78)\cdots(2^i-3\quad 2^i-2)(2^i-1\quad 2^i).$$

Let

$$z' = (13)(24)(56)(78)\cdots(2^i-3\quad 2^i-2)(2^i-1\quad 2^i)$$

so that $zz' = (14)(23)$, $z' \in A_{2^i}$ and $z \underset{A_{2^i}}{\sim} z'$. Thus the weak closure of $\{z\}$ in A_{2^i} contains $(14)(23)$. Consequently the group is equal to $A_{2^i}(2)$. Suppose now that $n = 2^{i_1} + \cdots + 2^{i_k}$ with $i_1 > \cdots > i_k$ and $k > 1$. Then $A_n(2) = (\Sigma_{2^{i_1}}(2) \times \cdots \times \Sigma_{2^{i_k}}(2)) \cap A_n$. If $\tau = \tau_1 \times \tau_2 \times \cdots \times \tau_k$ is contained in the center of $A_n(2)$, then $\tau_j \in Z(\Sigma_{2^{i_j}}(2))$. One can show that the weak closure of $\{\tau\}$ contains an element conjugate to $(12)(34)$. Thus this group must be $A_n(2)$. □

Next we consider the space $(B\Sigma_n)^\wedge_p$, where Σ_n is the symmetric group. For an odd prime p this group satisfies the hypothesis of Theorem 1. This follows from Proposition 2.2.

Theorem 2.3. (a) *If p is odd, then $(B\Sigma_n)^\wedge_p$ has no nontrivial retracts. The only nontrivial retract of $(B\Sigma_n)^\wedge_2$ for $n \geq 4$ is the space $B\mathbb{Z}/2$ up to homotopy.*

(b) *Let p be odd. If $n \equiv 0, 1 \bmod p$, then $[(B\Sigma_n)^\wedge_p, (BA_n)^\wedge_p] = 0$. If $n \not\equiv 0, 1 \bmod p$, then $(B\Sigma_n)^\wedge_p \simeq (BA_n)^\wedge_p$.*

(c) *The map $[B\mathbb{Z}/2, (BA_n)^\wedge_2] \to [(B\Sigma_n)^\wedge_2, (BA_n)^\wedge_2]$ induced by the projection $\Sigma_n \to \mathbb{Z}/2$ is bijective, where the kernel of this projection is A_n.*

Here we note related results about the unitary group $U(n)$ and the orthogonal group $O(n)$. On the level of classifying spaces we have the fibrations $BSU(n) \to BU(n) \to BS^1$ and $BSO(n) \to BO(n) \to B\mathbb{Z}/2$. From [10 and 11] one can observe the following

Theorem 2.4. (a) *The nontrivial p-local retracts of $BU(n)$ are p-equivalent to*

(i) BS^1 *if* $n \equiv 0 \bmod p$,

(ii) BS^1 *or* $BSU(n)$ *if* $n \not\equiv 0 \bmod p$.

(b) *The nontrivial p-local retracts of $BO(n)$ are p-equivalent to*

(i) *none* *if p is odd,*

(ii) $B\mathbb{Z}/2$ *if $p = 2$ and n is even,*

(iii) $B\mathbb{Z}/2$ *or* $BSO(n)$ *if $p = 2$ and n is odd.*

We also note that $BU(n) \underset{p}{\simeq} BS^1 \times BSU(n)$ when $n \not\equiv 0 \bmod p$, and that $BO(2k) \underset{p}{\simeq} BO(2k+1) \underset{p}{\simeq} BSO(2k+1)$ when p is odd. It is easy to see that $O(2k+1) \cong \mathbb{Z}/2 \times SO(2k+1)$ as groups for any k. Consequently $BO(2k+1) \simeq B\mathbb{Z}/2 \times BSO(2k+1)$ without localization. Finally, Theorem 2.3 implies that $(B\Sigma_n)^\wedge_p \simeq (BA_n)^\wedge_p$ if and only if $n \not\equiv 0, 1 \bmod p$.

We need the following lemma to prove part (b) of Theorem 2.3.

Lemma 2.5. *Let p be odd. The normalizers of the cyclic group $\mathbb{Z}/p\langle(12\cdots p)\rangle$ in Σ_p and in A_p are the following:*

$$N_{\Sigma_p}\mathbb{Z}/p = \mathbb{Z}/p \rtimes \mathbb{Z}/p-1, \quad N_{A_p}\mathbb{Z}/p = \mathbb{Z}/p \rtimes \mathbb{Z}\Big/\frac{p-1}{2}$$

where both $\mathbb{Z}/p-1$ and $\mathbb{Z}/\frac{p-1}{2}$ act freely on $(\mathbb{Z}/p)^*$.

Proof. Let b be a multiplicative generator of the unit group $(\mathbb{Z}/p)^* = \mathbb{Z}/p-1$. If $a = (12\cdots p)$, the bth power of a is another p-cycle. Hence we can find $g \in \Sigma_p$ such that $a^b = gag^{-1}$. If $S(\Sigma_p)$ denotes the set of all p-Sylow subgroups of Σ_p, then $|N_{\Sigma_p}\mathbb{Z}/p| = |\Sigma_p|/|S(\Sigma_p)| = p!/(p-2)! = p(p-1)$. Consequently

$$N_{\Sigma_p}\mathbb{Z}/p = \mathbb{Z}/p\langle a\rangle \rtimes \mathbb{Z}/p-1\langle g\rangle .$$

Similarly we see

$$N_{A_p}\mathbb{Z}/p = \mathbb{Z}/p\langle a\rangle \rtimes \mathbb{Z}\Big/\frac{p-1}{2}\langle g^2\rangle . \quad \square$$

Proof of Theorem 2.3. (a) For an odd prime p the desired result is obtained since Σ_n satisfies the hypothesis of Theorem 1 at p. It remains to show that if X is a nontrivial retract of $(B\Sigma_n)^\wedge_2$ for $n \geq 4$, then X is homotopy equivalent to $B\mathbb{Z}/2$. Suppose $X \underset{r}{\overset{i}{\rightleftarrows}} (B\Sigma_n)^\wedge_2$ with $r \circ i \simeq 1_X$. Let $f = i \circ r$. First we claim $f|BA_n \simeq 0$. Let P be a 2-Sylow subgroup of Σ_n. By Lemma 1.4 it suffices to show that if $f|BA_n \not\simeq 0$, then $f|BP \simeq B\rho$ for some injective homomorphism $\rho : P \to \Sigma_n$. If $f|BA_n \not\simeq 0$, Theorem 1 implies $\ker\rho \cap A_n = 1$. Recall that for a 2-Sylow subgroup Q of A_n we have $P = Q \rtimes \mathbb{Z}/2$. We notice that $|Q| \leq |\operatorname{Im}\rho| = |P|/|\ker\rho|$. Consequently $|\ker\rho| \leq 2$. An element τ of order 2 in Σ_n has the form $(a_1b_1)\cdots(a_kb_k)$ where a_i's and b_i's are mutually distinct. Suppose $\tau \in \ker\rho$. If $k = 1$, then Lemma 1.1 would imply that ρ is trivial, since transpositions generate the symmetric group. If $k \geq 2$, we consider a 2-Sylow subgroup containing the transpositions $(a_1b_1), (a_2b_2), \dots, (a_kb_k)$. One can see that at least two other elements in the 2-Sylow subgroup are conjugate to τ in Σ_n. This would imply $|\ker\rho| \geq 3$. Thus $\ker\rho = 1$. Since X is a nontrivial retract, this is a contradiction. Therefore $f|BA_n \simeq 0$.

We now consider the following commutative diagram

$$\begin{array}{ccccc} H^*(B\Sigma_n; \mathbb{F}_2) & \xleftarrow{\;r^*\;} & H^*(X; \mathbb{F}_2) & \xleftarrow{\;i^*\;} & H^*(B\Sigma_n; \mathbb{F}_2) \\ \downarrow & & & \swarrow{\scriptstyle (f|BA_n)^*} & \\ H^*(BA_n; \mathbb{F}_2) & & & & \end{array}$$

Notice that the image of $f^* = r^* \circ i^*$ is included in the kernel of $H^*(B\Sigma_n; \mathbb{F}_2) \to H^*(BA_n; \mathbb{F}_2)$. It is known that this kernel is the ideal generated by the generator w of $H^1(B\Sigma_n; \mathbb{F}_2)$, [17]. We claim $\operatorname{Im} f^* = \mathbb{F}_2[w]$. Since r^* is injective, we may identify $\operatorname{Im} f^*$ with $H^*(X; \mathbb{F}_2)$. If $w \notin H^1(X; \mathbb{F}_2)$, then $i^*(w) = 0$. This would imply $i^* = i^* \circ f^* = 0$. Thus $H^1(X; \mathbb{F}_2) = \mathbb{F}_2\langle w\rangle$ and hence $\mathbb{F}_2[w] \subset \operatorname{Im} f^*$. Next suppose y is an element of the set $H^*(X; \mathbb{F}_2) - \mathbb{F}_2[w]$ with minimal degree. We can write $y = wz$ for some $z \in H^*(B\Sigma_n; \mathbb{F}_2)$. Since $y = i^* \circ r^*(y) = i^*(w)i^*(z) = w \cdot i^*(z)$, it follows that $i^*(z) \in H^*(X; \mathbb{F}_2) - \mathbb{F}_2[w]$. This contradicts the minimality of the degree of y since $\deg w = 1$. Consequently $H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[w]$.

A section s for the group extension $A_n \to \Sigma_n \to \mathbb{Z}/2\langle t\rangle$ is given by $s(t) = (12)$. If $f|B\mathbb{Z}/2 \simeq 0$, then Lemma 1.1 implies $f = 0$. Thus $f|B\mathbb{Z}/2 \not\simeq 0$. It follows that the retract X is 2-equivalent to $B\mathbb{Z}/2$. Any retract of a p-complete space is p-complete. We now conclude that X is homotopy equivalent to $B\mathbb{Z}/2$.

(b) Suppose $f : (B\Sigma_n)^\wedge_p \to (BA_n)^\wedge_p$ is a nonzero map. Let D be a p-Sylow subgroup of Σ_n containing $E = \prod_{i=0}^{[(n-p)/p]} \mathbb{Z}/p\langle\sigma_i\rangle$ with $\sigma_i = (ip + 1 \cdots ip + p)$. If $f|BD \simeq B\rho'$, Theorem 1 says that ρ' is injective, since p is odd. Considering the conjugation by an element of A_n, we may assume $\rho'(D) = D$. Recall here that an element of order p is a product of distinct p-cycles. If $i : A_n \to \Sigma_n$ is the inclusion, the map $f \circ Bi$ is a homotopy equivalence. Using Lemma 1.1 one can find a nonnegative integer k such that $(f \circ Bi)^k \circ f|_{BD} \simeq B\rho$ where ρ is an automorphism which sends the p-cycles to the p-cycles. Let $e_i = \rho(\sigma_i)$ for $0 \le i \le [\frac{n-p}{p}]$. According to Lemma 2.5 there is $g \in N_{\Sigma_p}\mathbb{Z}/p$ such that $g\sigma_0 g^{-1} = \sigma_0^b$ where b is a multiplicative generator of $(\mathbb{Z}/p)^*$. If $\hat{g} = g \times 1 \times \cdots \times 1 \in \Sigma_p \times \Sigma_p \times \cdots \times \Sigma_p \subset \Sigma_n$, then $\rho|_E = \rho|_E \circ C_{\hat{g}}$ in $\mathrm{Rep}(E, A_n)$ since $BC_{\hat{g}} \simeq 1_{B\Sigma_n}$. Consequently there is $a \in A_n$ such that $C_a(e_0) = e_0^b$ and $C_a(e_i) = e_i$ for $1 \le i \le [\frac{n-p}{p}]$. We notice that $a \in N_{\Sigma_p}\mathbb{Z}/p \times \mathbb{Z}p \times \cdots \times \mathbb{Z}/p \times \Sigma_{n-p\cdot[n/p]}$ since the centralizer of $\mathbb{Z}/p$ in Σ_p is $\mathbb{Z}/p$ itself. If $n \equiv 0, 1 \mod p$, this would imply that there is $a' \in N_{A_p}\mathbb{Z}/p$ such that $a'\sigma_0(a')^{-1} = \sigma_0^b$. This contradicts Lemma 2.5. Therefore $[(B\Sigma_n)^\wedge_p, (BA_n)^\wedge_p] = 0$ if $n \equiv 0, 1 \mod p$.

Next, if $n \not\equiv 0, 1 \mod p$, then $\mathbb{Z}/2 = \Sigma_n/A_n$ acts trivially on a p-Sylow subgroup of A_n. By [3, p. 258] one can show that the map $H^*(B\Sigma_n; \mathbb{F}_p) \xrightarrow{(Bi)^*} H^*(BA_n; \mathbb{F}_p)$ is an isomorphism. Therefore $(B\Sigma_n)^\wedge_p \simeq (BA_n)^\wedge_p$.

(c) Let $f : (B\Sigma_n)^\wedge_2 \to (BA_n)^\wedge_2$. If $f \circ (Bi)^\wedge_2$ is a nonzero self-map of $(BA_n)^\wedge_2$, Theorem 1 and Lemma 1.5 imply that this map is a homotopy equivalence. This would imply that $(BA_n)^\wedge_2$ is a retract of $(B\Sigma_n)^\wedge_2$. According to part (a), this is a contradiction. Thus $f \circ (Bi)^\wedge_2 = 0$. One can show that f factors through $B\mathbb{Z}/2$ since $\mathrm{map}((B\Sigma_n)^\wedge_2, (BA_n)^\wedge_2) \simeq \mathrm{map}_{\mathbb{Z}/2}(E\mathbb{Z}/2, \mathrm{map}(BA_n, (BA_n)^\wedge_2))$ and the map $\lambda\colon (BA_n)^\wedge_2 \to \mathrm{map}(BA_n, (BA_n)^\wedge_2)_0$ is weakly equivalent. This proves the induce map is onto. Notice next that the map $H^*(B\mathbb{Z}/2; \mathbb{F}_2) \to H^*(B\Sigma_n; \mathbb{F}_2)$ is induced by the projection is a monomorphism. By a result of Lannes [12] we can show the map $[B\mathbb{Z}/2, (BA_n)^\wedge_2] \to [(B\Sigma_n)^\wedge_2, (BA_n)^\wedge_2]$ is one-to-one. □

3. General linear groups

Notation. Let $e_{ij}(\alpha) \in GL(n, \mathbb{F}_q)$ denote the elementary matrix with entry $\alpha \in \mathbb{F}_q^*$ in the (i, j)th place. We make a convention that $e_{ij} = e_{ij}(\alpha)$ for $\alpha = 1$. Next $d_{ij}(\alpha) \in Mat_n(\mathbb{F}_q)$ denotes the $n \times n$ matrix with entries 0 except the (i, j)th entry α. Equivalently $d_{ij}(\alpha) = e_{ij}(\alpha) - I_n$. We write $d_{ij} = d_{ij}(\alpha)$ for $\alpha = 1$, $d_i(\alpha) = d_{ij}(\alpha)$ for $i = j$, and $d_i = d_i(\alpha)$ for $\alpha = 1$.

Lemma 3.1. *The unipotent subgroup U_n of $GL(n, \mathbb{F}_q)$, upper triangular matrices with all diagonal entries equal to* 1, *is generated by the elementary matrices* $\{e_{ii+1}(\alpha) | \alpha \in \mathbb{F}_q^*, 1 \le i \le n-1\}$.

Proof. Suppose H_n is the subgroup of U_n generated by the above elementary matrices. We will show $H_n = U_n$ by induction. If $n = 2$, it is clear that $H_2 = U_2$. Assume $n \ge 3$ and the result holds up to $n-1$. By the hypothesis of induction we see $e_{ij}(\alpha) \in H_n$ for any $\alpha \in \mathbb{F}_q^*$ unless $i = 1$ and $j = n$:

$$\begin{pmatrix} H_{n-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} U_{n-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ p & H_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & U_{n-1} \end{pmatrix}.$$

Notice here that $e_{1n-1}(\alpha) \cdot e_{n-1n}(1) \cdot e_{1n-1}(-\alpha) \cdot e_{n-1n}(-1) = e_{1n}(\alpha)$. Hence $e_{1n}(\alpha) \in H_n$ for any $\alpha \in \mathbb{F}_q^*$. Any element in U_n has the form $\left(\begin{smallmatrix} 1 & B \\ 0 & A \end{smallmatrix}\right)$ where $A \in U_{n-1}$. This matrix decomposes as follows

$$\begin{pmatrix} 1 & B \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & I_{n-1} \end{pmatrix}$$

where I_{n-1} is the identity matrix. Since $U_{n-1} = H_{n-1}$, it follows that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right) \in H_n$. If $B = (b_2, b_3, \dots, b_n)$, then

$$\begin{pmatrix} 1 & B \\ 0 & I_{n-1} \end{pmatrix} = \prod_{i=2}^{n} e_{1i}(b_i) \in H_n .$$

Therefore $\left(\begin{smallmatrix} 1 & B \\ 0 & A \end{smallmatrix}\right) \in H_n$ and hence $H_n = U_n$. □

Lemma 3.2. *Any two of elementary matrices in the unipotent group U_n are conjugate to each other in $GL(n, \mathbb{F}_q)$.*

Proof. We will show that any $e_{ij}(\alpha)$ is conjugate to e_{12}. If D is the diagonal matrix $d_1(\alpha) + \sum_{1=1}^{n} d_i$, then $De_{12}D^{-1} = e_{12}(\alpha)$. If T is the permutation $(1\ i)(2\ j)$, then $Te_{12}(\alpha)T^{-1} = e_{ij}(\alpha)$. □

Lemma 3.3. *The center of U_n is $\{e_{1n}(\alpha) | \alpha \in \mathbb{F}_q^*\} \cup \{I_n\}$.*

Proof. Notice that $Ae_{ij} = e_{ij}A$ if and only if $Ad_{ij} = d_{ij}A$ where $A \in U_n$. If $A = (a_{ij})$ we see $Ad_{ij} = \sum_{k=1}^{n} d_{kj}a_{ki}$ and $d_{ij}A = \sum_{k=1}^{n} d_{ik}a_{jk}$. Hence $Ad_{ij} = d_{ij}A$ if and only if $a_{ki} = 0$ for $1 \le i \le n-1$ and $1 \le k \le i-1$, and $a_{jk} = 0$ for $2 \le j \le n$ and $j+1 \le k \le n$. This implies the desired result. □

Proposition 3.4. *The general linear group $GL(n, \mathbb{F}_q)$ satisfies the hypothesis of Theorem 1 at p where q is a power of p.*

Proof. First we note that the unipotent group U_n is a p-Sylow subgroup π_p of $GL(n, \mathbb{F}_q)$ since q is a power of p. Lemma 3.3 shows that $z = e_{1n}(\alpha)$ for some $\alpha \in \mathbb{F}_q^*$. The desired result follows from Lemma 3.1 and Lemma 3.2. □

The following lemma is known.

Lemma 3.5. *Any element in the finite field $\mathbb{F}_q$ is written as the sum of two squares.*

Proposition 3.6. *The special linear group $SL(n, \mathbb{F}_q)$ satisfies the hypothesis of Theorem 1 at p where q is a power of p.*

Proof. If $n \ge 3$, Lemma 3.2 is true for $SL(n, \mathbb{F}_q)$. In the proof the diagonal matrix D would be $d_1(\alpha^{-1}) + d_2 + \cdots + d_{n-1} + d_n(\alpha)$ and we can find a suitable T in $SL(n, \mathbb{F}_q)$. The rest of the argument is the same. For $n = 2$, Lemma 3.2 is false. But, since

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 1 & x^2\alpha \\ 0 & 1 \end{pmatrix},$$

Lemma 3.5 implies the desired result. □

Since the kernel of the projection $SL(n, \mathbb{F}_q) \to PSL(n, \mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/(n, q-1)$, [2, p. 62] shows $(BSL(n, \mathbb{F}_q))_p^\wedge \simeq (BPSL(n, \mathbb{F}_q))_p^\wedge$. Next, we can show $(BSL(n, \mathbb{F}_q))_p^\wedge \simeq (BGL(n, \mathbb{F}_q))_p^\wedge$ if and only if $(n, q-1) = 1$. In fact, if

$A(\alpha) = e_{12}(\alpha) + \sum_{i=2}^{n-1} d_{i\,i+1}$, then $A(\alpha) \in U_n$ and all $A(\alpha)$'s are conjugate each other in $GL(n, \mathbb{F}_q)$. In $SL(n, \mathbb{F}_q)$, however, $A(1)$ is not conjugate to $A(\beta)$ unless β is the nth power of some element of $\mathbb{F}_q^*$. From Theorem 1 and Lemma 1.1 it follows that $[(BGL(n, \mathbb{F}_q))_p^\wedge, (BSL(n, \mathbb{F}_q))_p^\wedge] = 0$ if $(n, q-1) \neq 1$. If $(n, q-1) = 1$, we see $GL(n, \mathbb{F}_q) \cong SL(n, \mathbb{F}_q) \times \mathbb{F}_q^*$. Because a scalar multiple of the identity αI_n for $\alpha \in \mathbb{F}_q^*$ is contained in $SL(n, \mathbb{F}_q)$ only if $\alpha = 1$ in this case.

4. Symplectic groups

Notation. Let $s_{ij}(\alpha)$ denote the $n \times n$ matrix $d_{ij}(\alpha) + d_{ji}(\alpha)$ for $i \neq j$. For example, if $n = 3$, then

$$s_{12}(\alpha) = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad s_{13}(\alpha) = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}.$$

We write $s_{ij} = s_{ij}(\alpha)$ for $\alpha = 1$. Next $r_{ij}(\alpha)$ denotes $d_{ij}(\alpha) + d_{ji}(-\alpha) \in Mat_n(\mathbb{F}_q)$ for $i \neq j$. Likewise $r_{ij} = r_{ij}(\alpha)$ for $\alpha = 1$.

For $A \in GL(n, \mathbb{F}_q)$, let $[A]$ denote $2n \times 2n$ matrix

$$\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \in GL(2n, \mathbb{F}_q).$$

For $B \in Mat_n(\mathbb{F}_q)$, let

$$\langle B \rangle = \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \in GL(2n, \mathbb{F}_q).$$

Lemma 4.1.

(i) $$[A]\langle B \rangle [A]^{-1} = \langle AB^tA \rangle,$$

(ii) $$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \alpha^2 b_1 & \alpha\beta b_2 \\ \alpha\beta b_3 & \beta^2 b_4 \end{pmatrix}.$$

Lemma 4.2. *Suppose $B = (b_{ij}) \in Mat_n(\mathbb{F}_q)$ for $1 \leq i, j \leq n$. If $AB^tA = B$ for any $A \in U_n$, then $b_{ij} = 0$ except $(i, j) = (1, 1), (1, 2)$ and $(2, 1)$.*

Proof. Suppose $k + m = n$. The matrix B is partitioned into 9 submatrices

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

where, for example, B_{11} is a $(k-2) \times (k-2)$ matrix, B_{22} is a 2×2 matrix and B_{33} is an $m \times m$ matrix. If E is a 2×2 matrix and

$$A = \begin{pmatrix} I_{k-2} & & 0 \\ & E & \\ 0 & & I_m \end{pmatrix},$$

then

$$AB^tA = \begin{pmatrix} B_{11} & B_{12}{}^tE & B_{13} \\ EB_{21} & EB_{22}{}^tE & EB_{23} \\ B_{31} & B_{32}{}^tE & B_{33} \end{pmatrix}.$$

Suppose $E = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ with $a \neq 0$ and $B = (b_{ij})$. Then $B_{12}{}^tE = B_{12}$ and $B_{32}{}^tE = B_{32}$ imply $b_{ik} = 0$ for $1 \leq i \leq k-2, k+1 \leq i \leq n$. Similarly $b_{kj} = 0$ for $1 \leq j \leq k-2, k+1 \leq j \leq n$ since $EB_{21} = B_{21}$ and $EB_{23} = B_{23}$. Finally $EB_{22}{}^tE = B_{22}$ implies $b_{kk} = 0$ for $2 \leq k \leq n$. □

Proposition 4.3. *The symplectic group $Sp(2n, \mathbb{F}_q)$ with $(n, q) \neq (2, 2)$ satisfies the hypothesis of Theorem 1 at p where q is a power of p.*

Proof. The subgroup $Sp(2n, \mathbb{F}_q)$ of $GL(2n, \mathbb{F}_q)$ corresponding to the symplectic form $\sum_{i=1}^n (X_iY_{n+i} - X_{n+i}Y_i)$ consists of those matrices M with $MJ_1{}^tM = J_1$ where

$$J_1 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

A p-Sylow subgroup is given by the semidirect product

$$\pi_p = \{\langle B\rangle | {}^tB = B\} \rtimes \{[A] | A \in U_n\},$$

[8, p. 192].

The center of U_n acts faithfully on the abelian group of $\langle B\rangle$'s. Lemma 2.1 and Lemma 4.2 imply that the center $Z(\pi_p)$ is included in the set $\{\langle d_1(b_1) + s_{12}(b_2)\rangle | b_1, b_2 \in \mathbb{F}_q\}$. If q is odd, then $b_2 = 0$. We need to consider three cases.

Case 1. Suppose $z = \langle d_1(b_1)\rangle$ for some $b_1 \neq 0$. Lemma 1.1 and Lemma 4.1 show that if K denotes the weak closure of $\{z\}$ in π_p, then $\langle d_1(\alpha^2 b_1)\rangle \in K$ for any $\alpha \in \mathbb{F}_q$. Since $\langle d_1(\alpha^2 b_1)\rangle \cdot \langle d_1(\beta^2 b_1)\rangle = \langle d_1((\alpha^2 + \beta^2)b_1)\rangle \in K$ for any $\alpha, \beta \in \mathbb{F}_q$, Lemma 3.5 implies $\langle d_1(b)\rangle \in K$ for any $b \in \mathbb{F}_q$. Note here that if $A \in GL(n, \mathbb{F}_q)$, then $[A] \in Sp(2n, \mathbb{F}_q)$. So $[A]\langle d_1\rangle[A]^{-1} = \langle d_2\rangle \in K$ by Lemma 1.1, if $A = s_{12} + \sum_{i=3}^n d_i$. For $A' = d_1 + s_{12} + \sum_{i=3}^n d_i$, we see $[A']\langle d_1 + d_2\rangle[A']^{-1} = \langle d_1(2) + s_{12} + d_2\rangle \in K$. Consequently $\langle d_1(2) + s_{12} + d_2\rangle \cdot ((\langle d_1\rangle^{-1})^2 \cdot \langle d_2\rangle^{-1} = \langle s_{12}\rangle \in K$. The abelian group $\{\langle B\rangle | B = {}^tB\}$ is generated by $\langle d_1(b)\rangle$ and $\langle s_{12}\rangle$ together with their conjugacy classes in $Sp(2n, \mathbb{F}_q)$. Lemma 1.1 implies $\langle B\rangle \in K$ for any $\langle B\rangle \in \pi_p$. Next we can show that if $R = I_{2n} + r_{2\ n+2} - d_2 - d_{n+2}$, then $R \in Sp(2n, \mathbb{F}_q)$ and $R\langle s_{12}\rangle R^{-1} = [e_{12}]$. Consequently, using Lemma 1.1, Lemma 3.1, and Lemma 3.2, we can show $[A] \in K$ for any $[A] \in \pi_p$ and therefore $K = \pi_p$.

Case 2. Suppose $z = \langle s_{12}(b_2)\rangle$ for some $b_2 \neq 0$. It suffices to show $\langle d_1(b)\rangle \in K$ for some $b \neq 0$ so that the argument is reduced to Case 1. Taking $\alpha = b_2^{-1}$ and $\beta = 1$ in Lemma 4.1(ii) we see $\langle s_{12}\rangle \in K$. Since $[e_{12}]$ is conjugate to $\langle s_{12}\rangle$ in $Sp(2n, \mathbb{F}_q)$, we see $[e_{12}] \in K$. Notice that

$$\langle s_{12}\rangle \cdot [e_{12}] = \begin{pmatrix} e_{12} & d_1(-1) + s_{12} \\ 0 & {}^te_{12}^{-1} \end{pmatrix} \in K.$$

If $Q = \left(\begin{smallmatrix} I_n & 0 \\ d_2 & I_n \end{smallmatrix}\right)$, then $Q \in Sp(2n, \mathbb{F}_q)$ and $Q\langle s_{12}\rangle[e_{12}]Q^{-1} = \langle d_1(-1) + s_{12}\rangle$. Consequently $\langle s_{12}\rangle \cdot \langle d_1(-1) + s_{12}\rangle^{-1} = \langle d_1\rangle \in K$.

Case 3. Suppose $z = \langle d_1(b_1) + s_{12}(b_2)\rangle$ for some $b_1 \neq 0$ and $b_2 \neq 0$. Hence q is assumed to be even. Again, it suffices to show $\langle d_1(b)\rangle \in K$ for some $b \neq 0$ unless $(q, n) = (2, 2)$. If $A = s_{12} + \sum_{i=3}^n d_i$, then $[A]\langle d_1(b_1) + s_{12}(b_2)\rangle[A]^{-1} =$

$\langle d_2(b_1)+s_{12}(b_2)\rangle \in K$. Hence $\langle d_1(b_1)+s_{12}(b_2)\rangle \cdot \langle d_2(b_1)+s_{12}(b_2)\rangle^{-1} = \langle d_1(b_1)+d_2(-b_1)\rangle \in K$. If $A' = d_1(x) + \sum_{i=2}^{n} d_i$ with $x \in \mathbb{F}_q^*$, then

$$[A']\langle d_1(b_1) + d_2(-b_1)\rangle[A']^{-1} \cdot \langle d_1(b_1) + d_2(-b_1)\rangle^{-1} = \langle d_1(b_1(x^2-1))\rangle \in K.$$

If $q \neq 2$, there is $x \in \mathbb{F}_q^*$ such that $x^2 \neq 1$. This implies $\langle d_1(b)\rangle \in K$ for some $b \neq 0$.

It remains to consider the case $q = 2$ and $n \geq 3$. Since $b_1 = 1$ and $b_2 = 1$ in this case, we can see $\langle d_1 + s_{12}\rangle$ and $\langle d_1 + d_2\rangle$ are contained in K. If $A = d_1 + d_2 + s_{12} + s_{23} + \sum_{i=4}^{n} d_i$, then $[A] \in Sp(2n, \mathbb{F}_2)$ for $n \geq 3$ and $[A]\langle d_1+d_2\rangle[A]^{-1} = \langle d_3+s_{13}+s_{23}\rangle \in K$. Since $\langle d_1+s_{12}\rangle$ is conjugate to $\langle d_3+s_{23}\rangle$ in $Sp(2n, \mathbb{F}_q)$, we see $\langle d_3+s_{23}\rangle \in K$ and hence $\langle d_3+s_{13}+s_{23}\rangle \cdot \langle d_3+s_{23}\rangle^{-1} = \langle s_{13}\rangle \in K$. Consequently $\langle s_{12}\rangle \in K$ and Case 2 shows $\langle d_1\rangle \in K$. This completes the proof. □

The kernel of the projection $Sp(2n, \mathbb{F}_q) \to PSp(2n, \mathbb{F}_q)$ is $\mathbb{Z}/2$ if q is odd and is trivial if q is even. Consequently $(BSp(2n, \mathbb{F}_q))_p^\wedge \simeq (BPSp(2n, \mathbb{F}_q))_p^\wedge$ if q is a power of p. The projective symplectic groups are all simple except for the cases $(n, q) = (1, 2), (1, 3), (2, 2)$. Note that $Sp(4, \mathbb{F}_2)$ is isomorphic to the symmetric group Σ_6.

5. Orthogonal groups

Proposition 5.1. *The orthogonal group $O(2n, \mathbb{F}_q)$ with $n \geq 3$ satisfies the hypothesis of Theorem* 1 *at p where q is a power of p.*

Proof. Recall that $O(2n, \mathbb{F}_q)$ can be regarded as the subgroup of $GL(2n, \mathbb{F}_q)$ which consists of those matrices that preserve the quadratic form $X_1X_{n+1} + X_2X_{n+2} + \cdots + X_nX_{2n}$ if q is even, or q is odd with n even or $4|q-1$. Because the discriminant of this quadratic form is equal to $(-1)^n/2^{2n}$, which is a square under the condition.

Assume first that q is even. Then a 2-Sylow subgroup is given by the semidirect product $\pi_2 = \{\langle B\rangle | {}^tB = B$ with $b_{ii} = 0$ for any $i\} \rtimes \{[A] | A \in U_n\}$, [8, p. 192]. For $n \geq 3$ the center of U_n acts faithfully on the abelian groups of $\langle B\rangle$'s. Using Lemma 2.1 and Lemma 4.2 we can show that $Z(\pi_2) = \{\langle s_{12}(b)\rangle | b \in \mathbb{F}_q\}$. Hence $z = \langle s_{12}(b)\rangle$ for some $b \neq 0$. Taking $b_1 = 0$, $b_2 = b_3$, and $b_4 = 0$ in Lemma 4.1(ii) we can show that any two elements of $\{\langle s_{12}(b)\rangle | b \neq 0\}$ are conjugate in $O(2n, \mathbb{F}_q)$. According to Lemma 1.1, $\langle s_{12}(b)\rangle \in K$ for any b where K is the weak closure of $\{z\}$ in π_2. For a fixed $b \in \mathbb{F}_q$, all the $\langle s_{ij}(b)\rangle$'s are conjugate to each other by the action of the $[\tau]$'s where τ is a permutation. Since the abelian group of $\langle B\rangle$ is generated by $\langle s_{ij}(b)\rangle$, it follows that $\langle B\rangle \in K$ for any $\langle B\rangle \in \pi_2$. Notice next that if T is the transposition interchanging X_2 and X_{n+2}, then $T \in O(2n, \mathbb{F}_q)$ and $T[e_{12}]T^{-1} = \langle s_{12}\rangle$. Consequently $[e_{12}] \in K$ and hence $\langle A\rangle \in K$ for any $A \in U_n$ by Lemma 1.1, Lemma 3.1, and Lemma 3.2. Therefore $K = \pi_2$.

Assume next that q is odd with n even or $4|q-1$. A p-Sylow subgroup is given by the semidirect product

$$\pi_p = \{\langle B\rangle | {}^tB = -B\} \rtimes \{[A] | A \in U_n\}.$$

The center of π_p is $\{\langle r_{12}(b)\rangle | b \in \mathbb{F}_q\}$. If T is the transposition interchanging X_2 and X_{n+2}, then $T \in O(2n, \mathbb{F}_q)$ and $T[e_{12}]T^{-1} = \langle r_{12}\rangle$. We can show $K = \pi_p$.

Next we consider the case that q is odd with n odd and $4 \nmid q-1$. The quadratic form $\sum_{i=1}^{n} X_i X_{n+i}$ is isomorphic to $\sum_{i=1}^{n-2} X_i X_{n+i} + X_{n-1}^2 - X_{2n-1}^2 + X_n^2 - X_{2n}^2$. The orthogonal group $O(2n-2; \mathbb{F}_q)$ can be regarded as the subgroup of $GL(2n-2, \mathbb{F}_q)$ which consists of those matrices that preserve the quadratic form $\sum_{i=1}^{n-2} X_i X_{n+i} + X_{n-1}^2 + X_n^2$. One can see that the injective map $O(2n-2, \mathbb{F}_q) \to O_-(2n, \mathbb{F}_q)$ sends a p-Sylow subgroup isomorphically into π_p. Namely $\langle B\rangle[A]$ is contained in the image if and only if $b_{i\,n-1} = a_{i\,n-1}$ for $1 \le i \le n-1$, $b_{in} = a_{in}$ for $1 \le i \le n-2$, and $b_{n-1\,n} = 0 = a_{n-1\,n}$. Here $A = (a_{ij})$ and $B = (b_{ij})$. The center of the group is $\{\langle r_{12}(b)\rangle | b \in \mathbb{F}_q\}$. When $n = 3$ and p is odd, we see $(BSL(4, \mathbb{F}_q))_p^\wedge \simeq (B\Omega(6, \mathbb{F}_q))_p^\wedge$ where $\Omega(6, q)$ is the commutator subgroup of $O(6, \mathbb{F}_q)$. Note that the index of $\Omega(6, \mathbb{F}_q)$ in $O(6, \mathbb{F}_q)$ is prime to p since p is odd. Lemma 1.1 together with the fact that $SL(4, \mathbb{F}_q)$ satisfies the desired result proves the case $n = 3$. An induction completes the proof. □

Let $\Omega(n, \mathbb{F}_q)$ denote the commutator subgroup of $O(n, \mathbb{F}_q)$. The kernel of the projection $\Omega(n, \mathbb{F}_q) \to P\Omega(n, \mathbb{F}_q)$ is at most $\mathbb{Z}/2$. Consequently, if p is odd, we see $(B\Omega(n, \mathbb{F}_q))_p^\wedge \simeq (BP\Omega(n, \mathbb{F}_q))_p^\wedge$. It is known that $P\Omega(2n, \mathbb{F}_q)$ is simple if $n \ge 3$. It is also known that if q is even, $Sp(2n, \mathbb{F}_q)$ is isomorphic to $O(2n+1, \mathbb{F}_q)$.

Proposition 5.2. *The group $\Omega(2n, \mathbb{F}_q)$ with $n \ge 3$ satisfies the hypothesis of Theorem 1 at p where q is a power of the odd prime p.*

Proof. Since q is odd, the commutator subgroup of $GL(n, \mathbb{F}_q)$ is $SL(n, \mathbb{F}_q)$. Recall that $[A] \in O(2n, \mathbb{F}_q)$ for any $A \in GL(n, \mathbb{F}_q)$ and that $[e_{12}]$ is conjugate to $\langle r_{12}\rangle$ in $O(2n, \mathbb{F}_q)$ when n is even or 4 divides $q-1$. One can see the p-Sylow subgroup π_p of $O(2n, \mathbb{F}_q)$ is also a p-Sylow subgroup of $\Omega(2n, \mathbb{F}_q)$ in this case. In the proof of Proposition 5.1, replace the transposition T by the permutation $(2\,n+2)(3\,n+3)$. We see the permutation is contained in $\Omega(2n, \mathbb{F}_q)$ since its spinor norm is 1. A similar argument completes the proof for this case. It is not hard to prove the other case. □

Proposition 5.3. *The orthogonal group $O(2n-1, \mathbb{F}_q)$ with $n \ge 3$ satisfies the hypothesis of Theorem 1 at p where q is a power of the odd prime p.*

Proof. First we need to find a suitable quadratic form for $O(2n-1, \mathbb{F}_q)$. The quadratic form $X_1 X_{n+1} + \cdots + X_n X_{2n}$ is isomorphic to $X_1 X_{n+1} + \cdots + X_{n-1} X_{2n-1} + X_n^2 - X_{2n}^2$. The group $O(2n-1, \mathbb{F}_q)$ can be regarded as the subgroup of $GL(2n-1, \mathbb{F}_q)$ which consists of those matrices that preserve the quadratic form $X_1 X_{n+1} + \cdots + X_{n-1} X_{2n-1} + X_n^2$. One can see that the injective map $O(2n-1, \mathbb{F}_q) \to O_\pm(2n, \mathbb{F}_q)$ sends a p-Sylow subgroup isomorphically into π_p. Namely $\langle B\rangle[A]$ is contained in the image if and only if $b_{in} = a_{in}$ for $1 \le i \le n-1$. The center of the group is $\{\langle r_{12}(b)\rangle | b \in \mathbb{F}_q\}$. When $n = 3$ and p is odd, we see $(BSp(4, \mathbb{F}_q))_p^\wedge \simeq (B\Omega(5, \mathbb{F}_q))_p^\wedge$. Note that the index of $\Omega(5, \mathbb{F}_q)$ in $O(5, \mathbb{F}_q)$ is prime to p since p is odd. Lemma 1.1 together with the result about $Sp(4, \mathbb{F}_q)$ proves the desired result for $n = 3$. An induction completes the proof. □

One can show that the analogous result holds for $\Omega(2n-1, \mathbb{F}_q)$ with $n \ge 3$.

6. Unitary groups

Notation. Let $t_{ij}(\alpha)$ denote the $n \times n$ matrix $d_{ij}(\alpha) = d_{ji}(-\alpha^q)$ for $i \neq j$ and $\alpha \in \mathbb{F}_{q^2}$. We write $t_{ij} = t_{ij}(\alpha)$ for $\alpha = 1$.

For $M = (m_{ij}) \in Mat_n(\mathbb{F}_{q^2})$, let $M^{(q)} = (m_{ij}^q) \in Mat_n(\mathbb{F}_{q^2})$. Let $M^* = {}^tM^{(q)}$. For example, if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$M^* = \begin{pmatrix} a^q & c^q \\ b^q & d^q \end{pmatrix}.$$

For $A \in GL(n, \mathbb{F}_{q^2})$, let $[A]$ denote

$$\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix} \in GL(2n, \mathbb{F}_{q^2}).$$

For $B \in Mat_n(\mathbb{F}_{q^2})$ let $\langle B \rangle = \left(\begin{smallmatrix} I_n & B \\ 0 & I_n \end{smallmatrix}\right) \in GL(2n, \mathbb{F}_{q^2})$.

Lemma 6.1.

(i) $$[A]\langle B\rangle[A]^{-1} = \langle ABA^* \rangle,$$

(ii) $$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\begin{pmatrix} \alpha^q & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha^{q+1}b_1 & \alpha b_2 \\ \alpha^q b_3 & b_4 \end{pmatrix}.$$

Lemma 6.2. *If $b \in \mathbb{F}_{q^2}$ and $b^{q-1} + 1 = 0$, then $\{\alpha^{q+1}b | \alpha \in \mathbb{F}_{q^2}\} = \{x \in \mathbb{F}_{q^2} | x^q + x = 0\}$.*

Proof. First we show $\alpha^{q+1}b$ is a solution of the equation $X^q + X = 0$. Since $\alpha^{q^2} = \alpha$ and $b^q + b = 0$, it follows that $(\alpha^{q+1}b)^q + \alpha^{q+1}b = \alpha^{q^2+q}b^q + \alpha^{q+1}b = \alpha^{q+1}(b^q + b) = 0$.

It remains to show that if $c^{q-1} + 1 = 0$, then $\frac{c}{b} = \alpha^{q+1}$ for some $\alpha \in \mathbb{F}_{q^2}$. Recall that $\mathbb{F}_{q^2}^*$ is isomorphic to the cyclic group $\mathbb{Z}/q^2 - 1$. Suppose a is a generator of this group. Then $b = a^k$ and $c = a^m$ for suitable k and m. If q is even, then $b^{q-1} = 1 = c^{q-1}$ and hence $k(q-1) = s(q^2-1)$ and $m(q-1) = t(q^2-1)$ for some $s, t \in \mathbb{Z}$. Consequently $m - k = (t-s)(q+1)$. Thus $\frac{c}{b} = a^{m-k} = (a^{t-s})^{q+1}$. In the case q is odd we have $k(q-1) = (q^2-1)/2 + s(q^2-1)$ and $m(q-1) = (q^2-1)/2 + t(q^2-1)$ for some $s, t \in \mathbb{Z}$, since $b^{q-1} = -1 = c^{q-1}$. Hence $m - k = (t-s)(q+1)$ and therefore $\frac{c}{b} = (a^{t-s})^{q+1}$. □

Proposition 6.3. *The unitary group $U(2n, \mathbb{F}_{q^2})$ satisfies the hypothesis of Theorem 1 at p where q is a power of p.*

Proof. The subgroup $U(2n, \mathbb{F}_{q^2})$ of $GL(2n, \mathbb{F}_{q^2})$ corresponding to the Hermitian form $\sum_{i=1}^n (X_i Y_{n+i}^q + X_{n+i} Y_i^q)$ consists of those matrices M with $MJ_0M^* = J_0$ where $M^* = {}^tM^{(q)}$ and $J_0 = \left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$. The semidirect product

$$\pi_p = \{\langle B \rangle | {}^tB = -B^{(q)}\} \rtimes \{[A] | A \in U_n\}$$

is a p-Sylow subgroup of $U(2n, \mathbb{F}_{q^2})$, [8, p. 192]. First we will show that $Z(\pi_p) = \{\langle d_1(b)\rangle | b \in \mathbb{F}_{q^2} \text{ with } b^q + b = 0\}$. Along the line of the proof of Lemma 4.2 we can show that if $\langle B \rangle$ is contained in the center $Z(\pi_p)$, then $B = d_1(b_1) + t_{12}(b_2)$ for some $b_1, b_2 \in \mathbb{F}_{q^2}$. We note that

$$[e_{12}(a)]\langle d_1(b_1) + t_{12}(b_2)\rangle[e_{12}(a)]^{-1} = \langle d_1(b_1 - ab_2^q + a^q b_2) + t_{12}(b_2)\rangle.$$

Hence $a^q b_2 - a b_2^q = 0$ for any $a \in \mathbb{F}_{q^2}$. Notice that the equation $b_2 X^q - b_2^q X = 0$ has at most q roots in $\mathbb{F}_{q^2}$ if $b_2 \neq 0$. Thus $b_2 = 0$. Since the action of $Z(U_n)$ on the abelian groups of $\langle B\rangle$'s is faithful, one can show the desired result.

Let K be the weak closure of $\{z\}$ in π_p where $z = \langle d_1(b)\rangle$ for some $b \in \mathbb{F}_{q^2}$ with $b^{q-1} + 1 = 0$. Lemma 6.2 together with Lemma 6.1 implies that $\langle d_1(b)\rangle \in K$ for any $b \in \mathbb{F}_{q^2}$ with $b^q + b = 0$. By an argument analogous to a part of the proof of Case 1 for $Sp(2n, \mathbb{F}_q)$ in §4, one can show $\langle s_{12}(b)\rangle \in K$ for such $b \in \mathbb{F}_{q^2}$. Since $b^q + b = 0$, it follows that $\langle s_{12}(b)\rangle = \langle t_{12}(b)\rangle$. Lemma 6.1 implies $\langle t_{12}(b)\rangle \in K$ for any $b \in \mathbb{F}_{q^2}$. The abelian group $\{\langle B\rangle | {}^t B = -B^{(q)}\}$ is generated by $\langle d_1(b)\rangle$ with $b^q + b = 0$ and $\langle t_{12}(b)\rangle$ for $b \in \mathbb{F}_{q^2}$ together with their conjugacy classes. Consequently $\langle B\rangle \in K$ for any $\langle B\rangle \in \pi_p$. Notice here that $t_{12} = r_{12}$. Hence, if T is the transposition interchanging X_2 and X_{n+2}, then $T \in U(2n, \mathbb{F}_{q^2})$ and $T[e_{12}]T^{-1} = \langle t_{12}\rangle$. Thus $[e_{12}] \in K$. This implies $[A] \in K$ for any $A \in U_n$. Consequently $K = \pi_p$. □

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